Math 122
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Spring 2018
Exam 2

Name: Solutions
March 23rd, 2018
Time Limit: 50 minutes

This exam contains 8 pages (including this cover page) and 15 questions.
The total number of marks is 150. You have 50 minutes to complete the exam.

Read each question carefully. When specified, you must show **all** *necessary* work to receive full credit.

No super fancy calculators/phone/smartwatch allowed under any circumstances. Place these items in your bag, out of reach. Cheating of any kind will not be tolerated and will result in a grade of zero. A regular calculator is allowed.

Question	Marks	Score	Question	Marks	Score
1	5		9	5	
2	5		10	5	
3	5		11	20	
4	5		12	20	
5	5		13	20	
6	5		14	20	
7	5		15	20	
8	5		Total	150	

Total Possible	Your Current Score	Your Current Percentage	Your Current Grade
570			

For questions 1-3, determine if the statement is true or false. By true we mean *always* true.

1. (5 marks) If $x = a$ is a stationary point of $f(x)$, then $f''(a) = 0$.

   A. True B. False

2. (5 marks) $\frac{d}{dx}\frac{u(x)}{v(x)} = u'(x)v(x)^{-1} - u(x)v'(x)v(x)^{-2}$.

   A. True B. False

3. (5 marks) If $f(x)$ is defined on the interval $a \leq x \leq b$, then the global extrema of $f(x)$ on $[a, b]$ occur only at the critical points of $f(x)$ on $[a, b]$.

   A. True B. False

For questions 4-7, fill in the blanks.

4. (5 marks) If $f(x)$ has a local maximum at $x = a$, then $f'(a)$ $=0$ and $f''(a)$ <0.

5. (5 marks) If $f''(a) = 0$, then $x = a$ is a point of inflection of $f(x)$.

6. (5 marks) The function $f(x) = x^2 - e^x$ is concave up at $x = 0$ and concave down at $x = 1$.

   $f'(x) = 2x - e^x$ $f''(0) = 2 - 1 = 1 > 0$
   $f''(x) = 2 - e^x$ $f''(1) = 2 - e < 0$

7. (5 marks) Complete the table below.

$\frac{d}{dx}(c) =$ 0	$\frac{d}{dx}(x^n) =$ nx^{n-1}
$\frac{d}{dx}(a^x) =$ $\ln(a)a^x$	$\frac{d}{dx}(\ln(x)) =$ $1/x$

For questions 8-10, choose the best answer. There is only one correct answer but you may choose up to *two*. If you choose two and one of the answers is correct, you will receive 3 marks.

8. (5 marks) Which of the following could be the function $f(x)$ with $f'(x) = 3x^2 + 3e^{-3x}$?

   A. $f(x) = x^3 - e^{-3x} + 2$

   B. $f(x) = 3x^3 + 3e^{3x} - 1$

   C. $f(x) = x^2 e^{-3x}$

   D. $f(x) = x^3 + e^{3x} - 1$

9. (5 marks) Suppose $f(3) = -8$, $f'(3) = 3$, $g(3) = 17$ and $g'(3) = -4$. What is $\frac{d}{dx} f(x)g(x)$ at $x = 3$?

   A. -48

   B. -12

   C. 83

   D. 92

10. (5 marks) Which of the following is the tangent line to the function $f(x) = x^2 + 1$ at the point $(2, 5)$.

   A. $y = x + 3$

   B. $y = 2x + 1$

   C. $y = 3x - 1$

   D. $y = 4x - 3$

For questions 11-15, show **all** *necessary* work to receive full credit. Please circle or box your final answer. If you cannot complete a problem but can write down what you want to do, and this is correct, you can still receive partial credit. Don't leave anything blank!

11. (a) (4 marks) Differentiate $y = 3x + x^2$, with respect to x.

$$\boxed{y' = 3 + 2x}$$

(b) (6 marks) Differentiate $y = (5 + x)^{-1}$, with respect to x.

Chain Rule

$u(x) = x^{-1}$ | $v(x) = 5 + x$

$u'(x) = -x^{-2}$ | $v'(x) = 1$

$y = u(v(x))$

$y' = v'(x)u'(v(x))$

$= 1 \cdot (-(5+x)^{-2})$

$$\boxed{y' = -(5+x)^{-2}}$$

(c) (6 marks) Differentiate $f(x) = \dfrac{3x + x^2}{5 + x}$, with respect to x.

Product rule

$f(x) = (3x + x^2)(5+x)^{-1}$

or use quotient rule

$u(x) = 3x + x^2$ | $v(x) = (5+x)^{-1}$

$u'(x) = 3 + 2x$ | $v'(x) = -(5+x)^{-2}$

$f(x) = u(x)v(x)$

$f'(x) = u'(x)v(x) + u(x)v'(x)$

$$\boxed{f'(x) = (3+2x)(5+x)^{-1} - (3x+x^2)(5+x)^{-2}}$$

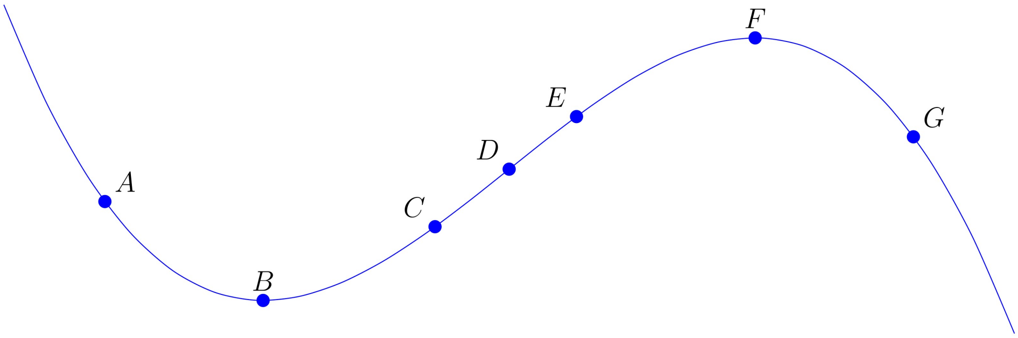
(d) (4 marks) Is $f(x)$ increasing, decreasing or stationary at $x = 2$? (circle *one*)

Increasing Decreasing Stationary

$f'(2) = (3+4)(5+2)^{-1} - (6+4)(5+2)^{-2}$

$= 7/7 - 10/49 = 1 - 10/49 = 39/49 > 0$

12.



Consider the graph of the function $f(x)$ above.

(a) (10 marks) Determine if each of the following quantities are positive, negative or zero. (circle *one*)

i. The derivative at A

Positive Negative Zero

ii. The second derivative at A

Positive Negative Zero

iii. The second derivative at D

Positive Negative Zero

(b) (10 marks) Find the *single* point that matches the following descriptions. (circle *one*)

i. $f'(x) > 0$, $f''(x) > 0$

A B C D E F G

ii. $f'(x) < 0$, $f''(x) < 0$

A B C D E F G

iii. $f'(x) > 0$, $f''(x) < 0$

A B C D E F G

13. The quantity of a drug in the bloodstream t hours after a tablet is swallowed is given, in mg, by the equation $q(t) = 36\left(e^{-t} - e^{-2t}\right)$.

(a) (4 marks) How much of the drug is in the bloodstream at time $t = 0$?

$$q(0) = 36(e^{-0} - e^{-0}) = 36(1 - 1) = \boxed{0}$$

(b) (12 marks) At what time is the quantity of drug in the bloodstream the highest?

$$q'(t) = 36(-e^{-t} + 2e^{-2t}) = 0$$

$$\implies -e^{-t} + 2e^{-2t} = 0$$

$$\implies 2e^{-2t} = e^{-t} \qquad (\times e^{2t})$$

$$\implies 2 = e^{t}$$

$$\implies \boxed{\ln(2) = t}$$

or $0.693 = t$

(c) (4 marks) What is the maximum quantity of the drug in the bloodstream?

* Plug in t from (b)

$$q(\ln(2)) = 36(e^{-\ln(2)} - e^{-2\ln(2)})$$

$$= 36(1/2 - 1/4) = 36(1/4) = \boxed{9\text{mg}}$$

14. At a price of \$80 for a half-day trip, a white-water rafting company attracts 300 customers. Every \$1 increase in price attracts 6 less customers.

(a) (5 marks) Find an equation relating the price of the ticket to the demand. That is, find an equation $q(p)$ that gives the quantity, q, of tickets sold in terms of the price, p. *(Hint: It's linear.)*

$$q - 300 = \frac{-6}{1}(p - 80)$$

$$\Longrightarrow \quad q = -6(p-80)+300 \quad \Longrightarrow \quad \boxed{q = 780 - 6p}$$

(b) (5 marks) Express the revenue generated by this company as a function of *price*.

$$R(p) = pq = p(780-6p)$$

$$\Longrightarrow \quad \boxed{R(p) = 780p - 6p^2}$$

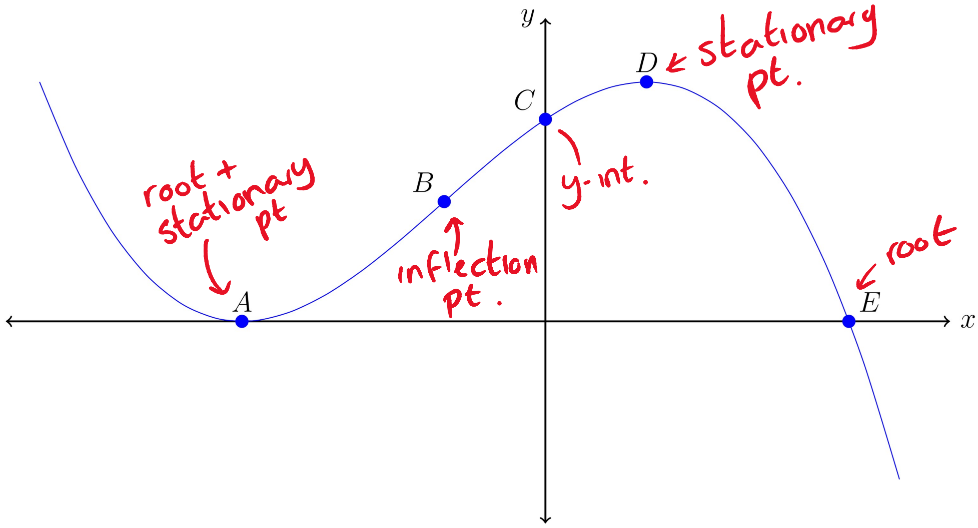
(c) (10 marks) Find the price that the company should sell each ticket for to maximise revenue.

$$R'(p) = 780 - 12p = 0$$

$$\Longrightarrow \quad 780 = 12p$$

$$\Longrightarrow \quad p = \frac{780}{12} = \boxed{\$65}$$

15. (20 marks)



The graph above is of the function $f(x) = -x^3 - 3x^2 + 9x + 27$. Use your knowledge of functions, the derivative and the second derivative to find the coordinates of each of the labelled points. *(Hint: You can factor $f(x)$ by grouping.)*

Point	A	B	C	D	E
Coordinates (x, y)	$(-3,0)$	$(-1,16)$	$(0,27)$	$(1,32)$	$(3,0)$

$$f(x) = -x^3 - 3x^2 + 9x + 27$$

$$f(0) = -0 - 0 + 0 + 27 = 27 \implies C = (0,27)$$

$$f(x) = -x^2(x+3) + 9(x+3)$$

$$= (9 - x^2)(x+3)$$

$$= (3-x)(3+x)(x+3) = 0 \implies A = (-3, f(-3))$$

$$E = (3, f(3))$$

$$f'(x) = -3x^2 - 6x + 9 = -3(x^2 + 2x - 3)$$

$$= -3(x+3)(x-1) = 0 \implies D = (1, f(1))$$

$$f''(x) = -6x - 6 = 0 \implies B = (-1, f(-1))$$